2012

Weighted Generalized Beta Distribution of the Second Kind

Yuan Ye
University of Houston

Broderick O. Oluyede
Georgia Southern University, boluyede@georgiasouthern.edu

Marvis Pararai
Indiana University of Pennsylvania

Recommended Citation

Ye, Yuan, Broderick O. Oluyede, Marvis Pararai. 2012. "Weighted Generalized Beta Distribution of the Second Kind." *Journal of Statistical and Econometric Methods*, 1 (1): 13-31. source: http://www.scienpress.com/journal_focus.asp?main_id=68&Sub_id=IV&Issue=199
https://digitalcommons.georgiasouthern.edu/math-sci-facpubs/143

Journal of Statistical and Econometric Methods, vol.1, no.1, 2012, 13-31
ISSN: 2241-0384 (print), 2241-0376 (online)
International Scientific Press, 2012

Weighted Generalized Beta Distribution of the Second Kind and Related Distributions

Yuan Ye[1], Broderick O. Oluyede[2] and Mavis Pararai[3]

Abstract

In this paper, a new class of weighted generalized beta distribution of the second kind (WGB2) is presented. The construction makes use of the "conservability approach" which includes the size or length-biased distribution as a special case. The class of WGB2 is used as descriptive models for the distribution of income. The results that are presented generalizes the generalized beta distribution of second kind (GB2). The properties of these distributions including behavior of hazard functions, moments, variance, coefficients of variation, skewness and kurtosis are obtained. The moments of other weighted distributions that are related to WGB2 are obtained. Other important properties including entropy (generalized and beta) which are measures of the uncertainty in this class of distributions are derived and studied.

Mathematics Subject Classification : 62E15
Keywords: GB2, WGB2, Moments, Generalized entropy, Beta entropy

1 Introduction

The generalized beta distribution of the second kind (GB2) is a very flexible four-parameter distribution. It captures the characteristics of income distri-

[1] Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, e-mail: yy00053@georgiasouthern.edu
[2] Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, e-mail: boluyede@georgiasouthern.edu
[3] Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA 15705, e-mail: pararaim@iup.edu

Article Info: *Received* : November 29, 2011. *Revised* : December 26, 2011
Published online : February 28, 2012

bution including skewness, peakness in low-middle range, and long right hand tail. This distribution, therefore provides good description of income distribution [8]. The GB2 also includes several other distributions as special or limiting cases, such as generalized gamma (GG), Dagum, beta of the second kind (B2), Singh-Maddala (SM), gamma, Weibull and exponential distributions.

The probability density function (pdf) of the generalized beta distribution of the second kind (GB2) is given by:

$$f_{GB2}(y;a,b,p,q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)[1+(\frac{y}{b})^a]^{p+q}} \quad \text{for } y>0, \text{ and } 0 \text{ otherwise}, \tag{1}$$

where a,p,q are shape parameters and b is scale parameter, $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function, and a,b,p,q are positive real values.

The $k^{th}-$ order moments of GB2 are given by [9]:

$$E_{GB2}\left(Y^k\right) = \frac{b^k\Gamma(p+\frac{k}{a})\Gamma(q-\frac{k}{a})}{\Gamma(p)\Gamma(q)}. \tag{2}$$

The moments exist if $-ap < k < aq$.

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. It adjusts the probabilities of actual occurrence of events to arrive at a specification of the probabilities when those events are recorded. Fisher [3] first introduced the concept of weighted distribution, in order to study the effect of ascertainment upon estimation of frequencies. Rao [14] unified concept of weighted distribution and use it to identify various sampling situations. Cox [1] and Zelen [18] introduced weighted distribution to present length biased sampling. Patil [12] used weighted distribution as stochastic models in the study of harvesting and predation. The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can also be seen in Nanda and Jain [10], Gupta and Keating [5], Oluyede [11] and in references therein.

Suppose Y is a non-negative random variable with its natural pdf $f(y;\theta)$, θ is a parameter, then the pdf of the weighted random variable Y^w is given by:

$$f^w(y;\theta,\beta) = \frac{w(y,\beta)f(y;\theta)}{\omega}, \tag{3}$$

where the weight function $w(y,\beta)$ is a non-negative function, that may depend on the parameter β, and $0 < \omega = E(w(Y,\beta)) < \infty$ is a normalizing constant.

In general, consider the weight function $w(y)$ defined as follows:

$$w(y) = y^k e^{ly} F^i(y) \overline{F}^j(y). \tag{4}$$

Setting $k = 0$; $k = j = i = 0$; $l = i = j = 0$; $k = l = 0$; $i \to i-1$; $j = n-i$; $k = l = i = 0$ and $k = l = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

This paper introduces a new class of weighted generalized beta distribution of the second kind (WGB2). The WGB2 is defined in Section 2, along with a discussion of related statistical properties. Section 3 considers the moments of WGB2 and several special cases. The generalized entropy and related economic indexes are presented in Section 4. In section 5, we present Renyi entropy for GB2 and WGB2 respectively.

2 Weighted Generalized Beta Distribution of the Second Kind

In particular, if we set $l = i = j = 0$ in the weight function (4), then we have $w(y) = y^k$. With the moments of GB2 in equation (2) we can obtain the corresponding pdf of weighted generalized beta distribution of the second kind (WGB2):

$$\begin{aligned} g_{WGB2}(y; a, b, p, q, k) &= \frac{y^k f(y; a, b, p, q)}{E(Y^k)} \\ &= \frac{y^k a y^{ap-1} \Gamma(p)\Gamma(q)}{b^{ap} B(p,q)[1 + (\frac{y}{b})^a]^{p+q} \cdot b^k \Gamma(p + \frac{k}{a})\Gamma(q - \frac{k}{a})} \\ &= \frac{a y^{ap+k-1}}{b^{ap+k} B(p + \frac{k}{a}, q - \frac{k}{a})[1 + (\frac{y}{b})^a]^{p+q}}, \end{aligned} \tag{5}$$

where $y > 0$, $a, b, p, q > 0$ and $-ap < k < aq$. WGB2 has one more parameter k compared to GB2.

The graphs of the pdf are given below:

Figure 1 depicts the pdf of WGB2 as the parameter k changes for representative values of the parameters $a, b, p, q : a = 1, b = 2, p = 4, q = 6$ for $k = 0, 1, 2, 3, 4$. We observe that: as the value of k increases, the "height" of the pdf becomes lower, and the pdf is more skewed right. The parameter k controls the shape and skewness of the density.

In order to further understand WGB2 with weight function $w(y) = y^k$, we discuss some related properties, including the cumulative distribution function

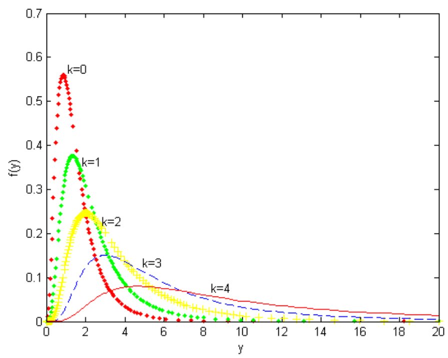

Figure 1: pdf of WGB2 ($a = 1, b = 2, p = 4, q = 6$)

(cdf), hazard function, monotonicity properties and elasticity.

The cdf of WGB2 is given by:

$$\begin{aligned} G_{WGB2}(y; a, b, p, q, k) &= \int_0^y \frac{a y^{ap+k-1}}{b^{ap+k} B(p+\frac{k}{a}, q-\frac{k}{a})[1+(\frac{y}{b})^a]^{p+q}} dy \\ &= 1 - I_{[1+(\frac{y}{b})^a]^{-1}}\left(p+\frac{k}{a}, q-\frac{k}{a}\right), \end{aligned} \tag{6}$$

where $I_x(\alpha, \beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)}$ is the incomplete beta function, $y > 0$, $a, b, p, q > 0$ and $-ap < k < aq$.

The graphs of the cdf of WGB2 are given below in Figure 2.

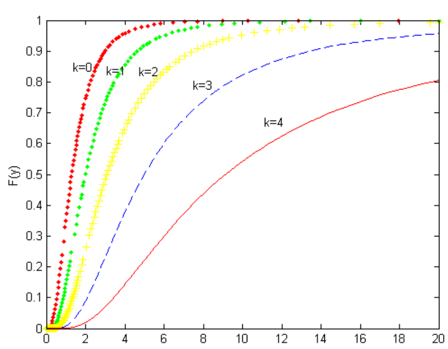

Figure 2: cdf of WGB2 ($a = 1, b = 2, p = 4, q = 6$)

Figure 2 depicts the cdf of WGB2 as the parameter k changes for representative values of the parameters a, b, p, q : $a = 1, b = 2, p = 4, q = 6$ for $k = 0, 1, 2, 3, 4$. We observe that as the value of k increases, the cdf increases slowly.

In Tables 1 and 2, some percentiles of WGB2 are presented. In particular, the 50^{th}, 75^{th}, 90^{th} and 95^{th} percentiles of WGB2 are given. The percentiles increases as b, p increases, and decreases as a, q increases with fixed k.

Table 1: Percentiles of WGB2 with $k = 1$

a	b	p	q	50th	75th	90th	95th
2.5	2.5	2.5	2.5	2.8986	3.7767	4.8844	5.7719
3				2.7696	3.4467	4.2578	4.8812
3.5				2.6949	3.2474	3.8859	4.3624
4				2.6477	3.1152	3.6411	4.0254
2.5	2.5	2.5	2.5	2.8986	3.7767	4.8844	5.7719
	3			3.4784	4.532	5.8613	6.9262
	3.5			4.0581	5.2873	6.8380	8.0806
	4			4.6378	6.0427	7.8150	9.2348
2.5	2.5	2.5	2.5	2.8986	3.7767	4.8844	5.7719
		3		3.1113	4.0262	5.1868	6.1196
		3.5		3.3043	4.2542	5.4644	6.439
		4		3.4818	4.4649	5.7217	6.7363
2.5	2.5	2.5	2.5	2.8986	3.7767	4.8844	5.7719
			3	2.6265	3.3533	4.2249	4.8932
			3.5	2.4265	3.0555	3.7831	4.3233
			4	2.2713	2.8314	3.4619	3.9189

The hazard function of WGB2 is given by:

$$\begin{aligned} h_{WGB2}(y; a, b, p, q, k) &= \frac{g_{WGB2}(y; a, b, p, q, k)}{\overline{G}_{WGB2}(y; a, b, p, q, k)} \\ &= \frac{g_{WGB2}(y; a, b, p, q, k)}{1 - G_{WGB2}(y; a, b, p, q, k)} \\ &= \frac{a y^{ap+k-1}[1 + (\frac{y}{b})^a]^{-(p+q)}}{b^{ap+k} B(p + \frac{k}{a}, q - \frac{k}{a}) I_{[1+(\frac{y}{b})^a]^{-1}}(p + \frac{k}{a}, q - \frac{k}{a})}, \end{aligned}$$

for $y > 0$, $a, b, p, q > 0$ and $-ap < k < aq$.

The graphs of the hazard functions are given below in Figure 3.

Table 2: Percentiles of WGB2 with $k = 2$

a	b	p	q	50th	75th	90th	95th
2.5	2.5	2.5	2.5	3.3966	4.5133	6.0133	7.281
3				3.0834	3.8815	4.8820	5.6804
3.5				2.9126	3.5359	4.2819	4.855
4				2.8084	3.3212	3.9151	4.3592
2.5	2.5	2.5	2.5	3.3966	4.5133	6.0133	7.281
	3			4.0759	5.416	7.2160	8.7372
	3.5			4.7552	6.3186	8.4187	10.1935
	4			5.4346	7.2213	9.6214	11.6495
2.5	2.5	2.5	2.5	3.3966	4.5133	6.0133	7.281
		3		3.6141	4.7791	6.3507	7.6818
		3.5		3.8136	5.0242	6.6628	8.053
		4		3.9987	5.2524	6.9541	8.3998
2.5	2.5	2.5	2.5	3.3966	4.5133	6.0133	7.281
			3	3.0045	3.8736	4.9623	5.8277
			3.5	2.7349	3.4591	4.3246	4.9849
			4	2.5343	3.1631	3.8892	4.4269

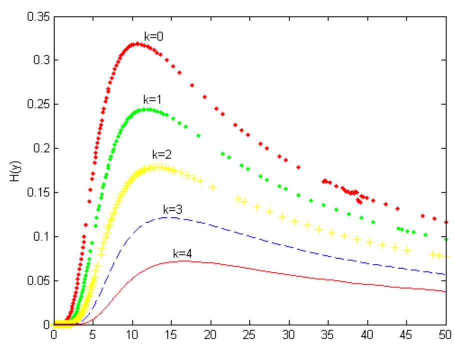

Figure 3: Hazard functions of WGB2

Next we study the monotonicity properties and discuss the income-shared elasticity of WGB2. In order discuss monotonicity of WGB2, we take the logarithm of its pdf:

$$ln(g_{_{WGB2}}(y;a,b,p,q,k)) = lnC + (ap+k-1)(lny - lnb) - (p+q)ln\left[1+\left(\frac{y}{b}\right)^a\right],$$

where C is a constant. Note that

$$\frac{\partial lng_{_{WGB2}}(y;a,b,p,q,k)}{\partial y} = \frac{(ap+k-1)b^a - (aq-k+1)y^a}{y(b^a+y^a)},$$

where $y>0$, $b,p,q>0$, and$-ap<k<aq$, so $aq-k+1>0$. It follows therefore that

$$\frac{\partial lng_{_{WGB2}}(y;a,b,p,q,k)}{\partial y} > 0 \Leftrightarrow y < b\left(\frac{ap+k-1}{aq-k+1}\right)^{\frac{1}{a}},$$

$$\frac{\partial lng_{_{WGB2}}(y;a,b,p,q,k)}{\partial y} = 0 \Leftrightarrow y = b\left(\frac{ap+k-1}{aq-k+1}\right)^{\frac{1}{a}},$$

$$\frac{\partial lng_{_{WGB2}}(y;a,b,p,q,k)}{\partial y} < 0 \Leftrightarrow y > b\left(\frac{ap+k-1}{aq-k+1}\right)^{\frac{1}{a}}.$$

The mode of WGB2 is $y_0 = b\left(\frac{ap+k-1}{aq-k+1}\right)^{\frac{1}{a}}$.

The income-share elasticity is defined as $\frac{-yg^{'}(y)}{g(y)}$, where $g^{'}(y) = \frac{dg(y)}{dy}$. See Esteban [2] for additional details. The derivative of $g_{_{WGB2}}(y;a,b,p,q,k)$ with respect to y is given by:

$$\begin{aligned}
g^{'}_{_{WGB2}}(y) &= \left[\frac{ay^{ap+k-1}}{b^{ap+k}B(p+\frac{k}{a},q-\frac{k}{a})[1+(\frac{y}{b})^a]^{p+q}}\right]^{'} \\
&= \frac{a}{bB(p+\frac{k}{a},q-\frac{k}{a})}\left[\left(\frac{y}{b}\right)^{ap+k-1}\left[1+\left(\frac{y}{b}\right)^a\right]^{-(p+q)}\right]^{'} \\
&= \frac{ay^{ap+k-2}[ap+k-1-(p+q)\frac{y^a}{b^a+y^a}]}{b^{ap+k}B(p+\frac{k}{a},q-\frac{k}{a})[1+(\frac{y}{b})^a]^{p+q}}.
\end{aligned}$$

The income-share elasticity of WGB2 is given by:

$$\begin{aligned}
\delta_{G_{_{WGB2}}}(y;a,b,p,q,k) &= \frac{-yg^{'}_{WGB2}(y;a,b,p,q,k)}{g_{WGB2}(y;a,b,p,q,k)} \\
&= (p+q)\frac{y^a}{b^a+y^a} - ap - k + 1. \qquad (7)
\end{aligned}$$

3 Moments of WGB2 and Related Special Cases

3.1 Moments

The non central (j^{th}) moment of WGB2 is given by:

$$\begin{aligned}
E_{G_{WGB2}}(Y^j) &= \int_0^\infty y^j g_{WGB2}(y)dy \\
&= \int_0^\infty \frac{y^j a y^{ap+k-1}}{b^{ap+k} B(p+\frac{k}{a}, q-\frac{k}{a})[1+(\frac{y}{b})^a]^{p+q}} dy \\
&= \frac{ab^{j-1}}{B(p+\frac{k}{a}, q-\frac{k}{a})} \int_0^\infty \left(\frac{y}{b}\right)^{ap+k+j-1} \left[1+\left(\frac{y}{b}\right)^a\right]^{-(p+q)} dy \\
&= \frac{ab^{j-1}}{B(p+\frac{k}{a}, q-\frac{k}{a})} \int_0^\infty \left[\left(\frac{y}{b}\right)^a\right]^{p+\frac{a}{k}+\frac{j}{a}-\frac{1}{a}} \left[1+\left(\frac{y}{b}\right)^a\right]^{-(p+q)} dy.
\end{aligned}$$

Let $(\frac{y}{b})^a = t$, then $y = bt^{\frac{1}{a}}, dy = \frac{b}{a} t^{\frac{1}{a}-1} dt$, and

$$\begin{aligned}
E_{G_{WGB2}}(Y^j) &= \frac{b^j}{B(p+\frac{k}{a}, q-\frac{k}{a})} \int_0^\infty t^{p+\frac{a}{k}+\frac{j}{a}-1}(1+t)^{-(p+q)} dt \\
&= \frac{b^j B(p+\frac{k}{a}+\frac{j}{a}, q-\frac{k}{a}-\frac{j}{a})}{B(p+\frac{k}{a}, q-\frac{k}{a})}. \qquad (8)
\end{aligned}$$

The mean and variance of the WGB2 distribution are given by:

$$\mu_{G_{WGB2}} = E_{G_{WGB2}}(Y) = \frac{bB(p+\frac{k}{a}+\frac{1}{a}, q-\frac{k}{a}-\frac{1}{a})}{B(p+\frac{k}{a}, q-\frac{k}{a})}, \qquad (9)$$

and

$$\begin{aligned}
Var_{G_{WGB2}}(Y) &= E_{G_{WGB2}}(Y^2) - (E_{G_{WGB2}}(Y))^2 \\
&= b^2 \left[\frac{B(p+\frac{k+2}{a}, q-\frac{k+2}{a})}{B(p+\frac{k}{a}, q-\frac{k}{a})} - \left(\frac{B(p+\frac{k+1}{a}, q-\frac{k+1}{a})}{B(p+\frac{k}{a}, q-\frac{k}{a})}\right)^2\right] \qquad (10)
\end{aligned}$$

respectively.

The coefficient of variation (CV) is given by:

$$CV = \frac{\sqrt{Var_{G_{WGB2}}(Y)}}{\mu_{G_{WGB2}}} = \sqrt{\frac{B(p+\frac{k+2}{a}, q-\frac{k+2}{a})B(p+\frac{k}{a}, q-\frac{k}{a})}{B^2(p+\frac{k+1}{a}, q-\frac{k+1}{a})} - 1}. \qquad (11)$$

Similarly, the coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by:

$$CS = E\left[\left(\frac{Y-\mu}{\sigma}\right)^3\right] = \frac{E[Y^3] - 3\mu E[Y^2] + 2\mu^3}{\sigma^3}, \qquad (12)$$

and

$$CK = E\left[\left(\frac{Y-\mu}{\sigma}\right)^4\right] = \frac{E[Y^3] - 4\mu E[Y^3] + 6\mu^2 E[Y^2] - 3\mu^4}{\sigma^4}, \tag{13}$$

where

$$\mu = \mu_{G_{WGB2}}, \quad \sigma = \sqrt{Var_{G_{WGB2}}(Y)}, \quad E[Y^2] = \frac{b^2 B(p + \frac{k}{a} + \frac{2}{a}, q - \frac{k}{a} - \frac{2}{a})}{B(p + \frac{k}{a}, q - \frac{k}{a})},$$

$$E[Y^3] = \frac{b^3 B(p + \frac{k}{a} + \frac{3}{a}, q - \frac{k}{a} - \frac{3}{a})}{B(p + \frac{k}{a}, q - \frac{k}{a})}, \quad E[Y^4] = \frac{b^4 B(p + \frac{k}{a} + \frac{4}{a}, q - \frac{k}{a} - \frac{4}{a})}{B(p + \frac{k}{a}, q - \frac{k}{a})}.$$

Since we have obtained the mode, mean, variance, CV, CS and CK of WGB2, we can set the values of the parameters a, b, p, q and compute the values of these quantities in Tables 3 and 4.

Table 3: The mode, mean, variance, CV, CS and CK of WGB2 when $k = 1$

a	b	p	q	mode	mean	variance	CV	CS	CK
2.5	2.5	2.5	2.5	2.5	3.187619	2.072506	0.451629	-13.398078	22.993346
3				2.5	2.954545	1.132674	0.360215	-21.542438	11.61182
3.5				2.5	2.824413	0.723843	0.301227	-32.416597	8.227137
4				2.5	2.743821	0.50716	0.259547	-46.503715	6.672292
2.5	2.5	2.5	2.5	2.5	3.187619	2.072506	0.451629	-13.398078	22.993346
	3			2.5	3.825143	2.984408	0.451629	-14.081444	22.99334
	3.5			2.5	4.462667	4.062111	0.451629	-14.569562	22.993346
	4			2.5	5.100191	5.305615	0.451629	-14.935651	22.993346
2.5	2.5	2.5	2.5	2.5	3.187619	2.072506	0.451629	-13.398078	22.993346
		3		2.6891	3.417945	2.263706	0.440195	-14.444185	23.727663
		3.5		2.8602	3.627291	2.45092	0.431601	-15.321162	24.322348
		4		3.0171	3.820056	2.634649	0.424905	-16.067381	24.813225
2.5	2.5	2.5	2.5	2.5	3.187619	2.072506	0.451629	-13.398078	22.993346
			3	2.3242	2.829373	1.287897	0.401098	-16.618157	11.383826
			3.5	2.1852	2.580454	0.91433	0.370557	-19.08715	7.988561
			4	2.0715	2.394085	0.702145	0.350005	-20.958808	6.438066

Table 4: The mode, mean, variance, CV, CS and CK of WGB2 when $k = 2$

a	b	p	q	mode	mean	variance	CV	CS	CK
2.5	2.5	2.5	2.5	2.8445	3.8378	3.9325	0.5167	-10.6613	137.0961
3				2.7339	3.3379	1.7017	0.3908	-18.6434	20.2011
3.5				2.6695	3.0807	0.9626	0.3185	-29.3114	11.2619
4				2.6286	2.9287	0.6268	0.2703	-43.1653	8.2368
2.5	2.5	2.5	2.5	2.8445	3.8378	3.9325	0.5167	-10.6613	137.0961
	3			3.4134	4.6054	5.6628	0.5167	-11.0601	137.0961
	3.5			3.9824	5.3729	7.7077	0.5167	-11.345	137.0961
	4			4.5513	6.1405	10.0672	0.5167	-11.5587	137.0961
2.5	2.5	2.5	2.5	2.8445	3.8378	3.9325	0.5167	-10.6613	137.0961
		3		3.0314	4.0802	4.305	0.5085	-11.1588	140.504
		3.5		3.2024	4.303	4.6694	0.5022	-11.5722	143.2584
		4		3.3607	4.5097	5.0268	0.4972	-11.9217	145.5293
2.5	2.5	2.5	2.5	2.8445	3.8378	3.9325	0.5167	-10.6613	137.0961
			3	2.6116	3.2846	1.9893	0.4294	-15.0613	19.4437
			3.5	2.4342	2.9348	1.2665	0.3835	-18.6267	10.6813
			4	2.2929	2.6874	0.9093	0.3548	-21.4601	7.7605

From the tables, we observe the following:
1) When $k = 1$, mode increases as p increases, decreases as q increases, and does not change as a, b increases; when $k = 2$, mode increases as b, p increases, decreases as a, q increases.
2) Mean, variance decreases as a, q increases, increases as b, p increases.
3) CV decreases as a, p, q increases, and does not change as b increases.
4) CS decreases as a, b, p, q increases.
5) CK decreases as a, q increases, increases as p increases, and does not change as b increases.

3.2 Special cases

WGB2 includes several other distributions as special or limiting cases, such as weighted generalized gamma (WGG), weighted beta of the second kind (WB2), weighted Singh-Maddala (WSM), weighted Dagum (WD), weighted gamma (WG), weighted Weibull (WW) and weighted exponential (WE) distributions.

We can also obtain the j^{th} moments of these distributions with the weight function $w(y) = y^k$.[4]

[4] In the special cases, one should consider the restrictions on the values of k and j.

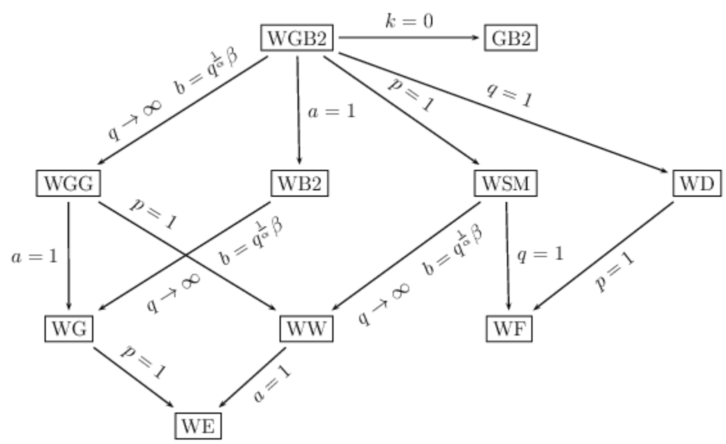

Figure 4: Graph Tree

- Weighted Singh-Maddala ($p = 1$)

$$E_{G_{WSM}}(Y^j) = \frac{b^j B(1 + \frac{k}{a} + \frac{j}{a}, q - \frac{k}{a} - \frac{j}{a})}{B(1 + \frac{k}{a}, q - \frac{k}{a})}. \tag{14}$$

- Weighted Dagum ($q = 1$)

$$E_{G_{WD}}(Y^j) = \frac{b^j B(p + \frac{k}{a} + \frac{j}{a}, 1 - \frac{k}{a} - \frac{j}{a})}{B(p + \frac{k}{a}, 1 - \frac{k}{a})}. \tag{15}$$

- Weighted Beta of the Second Kind ($a = 1$)

$$E_{G_{WB2}}(Y^j) = \frac{b^j B(p + k + j, q - k - j)}{B(p + k, q - j)}. \tag{16}$$

- Weighted Generalized Gamma ($b = q^{\frac{1}{\alpha}}\beta$ as $q \to \infty$)

$$E_{G_{WGG}}(Y^j) = \frac{\beta^j \Gamma(p + \frac{k}{a} + \frac{j}{a})}{\Gamma(p + \frac{k}{a})}. \tag{17}$$

- Weighted Fisk ($p = 1, q = 1$)

$$E_{G_{WF}}(Y^j) = \frac{b^j B(1 + \frac{k}{a} + \frac{j}{a}, 1 - \frac{k}{a} - \frac{j}{a})}{B(1 + \frac{k}{a}, 1 - \frac{k}{a})}. \tag{18}$$

- Weighted Gamma ($a = 1$, $b = q^{\frac{1}{\alpha}}\beta$ as $q \to \infty$)

$$E_{G_{WG}}(Y^j) = \frac{\beta^j \Gamma(p+k+j)}{\Gamma(p+k)}. \tag{19}$$

- Weighted Weibull ($p = 1$, $b = \beta q^{\frac{1}{a}}$ as $q \to \infty$)

$$E_{G_{WW}}(Y^j) = \frac{\beta^j (k+j)\Gamma(\frac{k}{a}+\frac{j}{a})}{k\Gamma(\frac{k}{a})}. \tag{20}$$

- Weighted Exponential ($a = p = 1$, $b = \beta q^{\frac{1}{a}}$ as $q \to \infty$)

$$E_{G_{WE}}(Y^j) = \frac{\beta^j (k+j)!}{k!}. \tag{21}$$

4 Generalized Entropy

Generalized entropy (GE) is widely used to measure inequality trends and differences. It is primarily used in income distribution. Kleiber and Kotz [7] derived Theil index for GB2 and Singh-Maddala model.

The generalized entropy (GE) $I(\alpha)$ is defined as:

$$I(\alpha) = \frac{v_\alpha \mu^{-\alpha} - 1}{\alpha(\alpha-1)}, \alpha \neq 0, \alpha \neq 1, \tag{22}$$

where $v_\alpha = \int y^\alpha dF(y)$, $\mu \equiv E(Y)$ is the mean, and $F(y)$ is the cumulative distribution function (cdf) of the random variable Y. The bottom-sensitive index is $I(-1)$, and the top-sensitive index is $I(2)$.

The mean logarithmic deviation (MLD) index is given by:

$$I(0) = lim_{\alpha\to 0} I(\alpha) = log\mu - v_0. \tag{23}$$

and Theil index is:

$$I(1) = lim_{\alpha\to 1} I(\alpha) = \frac{\mu}{v_1} - log\mu. \tag{24}$$

The generalized entropy of GB2 is given by Jenkins [6] as:

$$I(\alpha)=\frac{B(p+\frac{\alpha}{a},q-\frac{\alpha}{a})B^{-\alpha}(p+\frac{1}{a},q-\frac{1}{a})-B^{1-\alpha}(p,q)}{\alpha(\alpha-1)B^{1-\alpha}(p,q)},\alpha\neq 0,\alpha\neq 1,$$

with

$$I(0)=\Gamma\left(p+\frac{1}{a}\right)+\Gamma\left(q-\frac{1}{a}\right)-\Gamma(p)-\Gamma(q)-\frac{\psi(p)}{a}-\frac{\psi(q)}{a},$$

and

$$I(1)=\frac{\psi(p+\frac{1}{a})}{a}-\frac{\psi(q-\frac{1}{a})}{a}-\Gamma\left(p+\frac{1}{a}\right)-\Gamma\left(q-\frac{1}{a}\right)+\Gamma(p)+\Gamma(q).$$

From our previous discussions about WGB2, v_α and μ are given by:

$$v_\alpha=\frac{b^\alpha B(p+\frac{k}{a}+\frac{\alpha}{a},q-\frac{k}{a}-\frac{\alpha}{a})}{B(p+\frac{k}{a},q-\frac{k}{a})},\quad \text{and} \qquad \mu=\frac{bB(p+\frac{k}{a}+\frac{1}{a},q-\frac{k}{a}-\frac{1}{a})}{B(p+\frac{k}{a},q-\frac{k}{a})},$$

respectively.

Consequently, the generalized entropy of WGB2 is given by:

$$I(\alpha)=\frac{B(p+\frac{k}{a}+\frac{\alpha}{a},q-\frac{k}{a}-\frac{\alpha}{a})B^{-\alpha}(p+\frac{k}{a}+\frac{1}{a},q-\frac{k}{a}-\frac{1}{a})-B^{1-\alpha}(p+\frac{k}{a},q-\frac{k}{a})}{\alpha(\alpha-1)B^{1-\alpha}(p+\frac{k}{a},q-\frac{k}{a})}, \tag{25}$$

where $\alpha\neq 0$ and $\alpha\neq 1$. Note that $I(\alpha)$ does not depend on the scale parameter b.

When $\alpha=0$ or $\alpha=1$, set $m(\alpha)=v_\alpha\mu^{-\alpha}-1$, $n(\alpha)=\alpha(\alpha-1)$, then $I(\alpha)=\frac{m(\alpha)}{n(\alpha)}$. By L'Hopital's rule, we have $I(0)=-m^{'}(0)$, $I(1)=m^{'}(1)$,

$$m^{'}(\alpha)=(\mu^{-\alpha})^{'}v_\alpha+(\mu^{-\alpha})v_\alpha^{'},\quad (\mu^{-\alpha})^{'}=-\mu^{-\alpha}log\mu,$$

and

$$v_\alpha^{'}=v_\alpha[\frac{\psi(p+\frac{k}{a}+\frac{\alpha}{a})}{a}-\frac{\psi(q-\frac{k}{a}-\frac{\alpha}{a})}{a}+logb],\quad where \quad \psi(z)=\frac{\Gamma^{'}(z)}{\Gamma(z)}.$$

The MLD index and Theil index, I(0) and I(1) of WGB2 are:

$$I(0)=log\frac{B(p+\frac{k+1}{a},q-\frac{k+1}{a})}{B(p+\frac{k}{a},q-\frac{k}{a})}-\frac{\psi(p+\frac{k}{a})}{a}-\frac{\psi(q-\frac{k}{a})}{a}, \tag{26}$$

and

$$I(1) = \frac{\psi(p + \frac{k+1}{a})}{a} - \frac{\psi(q - \frac{k+1}{a})}{a} - log\frac{B(p + \frac{k+1}{a}, q - \frac{k+1}{a})}{B(p + \frac{k}{a}, q - \frac{k}{a})} \tag{27}$$

respectively. Note that I(0) and I(1) does not depend on the scale parameter b.

We select the values of the parameters a, p, q and compute the bottom-sensitive index (I(-1)), top-sensitive index (I(2)), mean logarithmic deviation (MLD) index (I(0)) and Theil index (I(1)) in Tables 5 and 6.

Table 5: Generalized entropy of WGB2 with $k = 1$

a	p	q	I(-1)	I(2)	I(0)	I(1)
2.5	2.5	2.5	0.0891	0.102	0.0841	0.2544
3			0.0594	0.0649	0.0572	0.2649
3.5			0.0426	0.0454	0.0414	0.2569
4			0.0322	0.0337	0.0315	0.2433
2.5	2.5	2.5	0.0891	0.102	0.0841	0.2544
	3		0.0826	0.0969	0.0791	0.2497
	3.5		0.0779	0.0931	0.0754	0.2464
	4		0.0745	0.0903	0.0726	0.2438
2.5	2.5	2.5	0.0891	0.102	0.0841	0.2544
		3	0.0759	0.0804	0.0712	0.5079
		3.5	0.0678	0.0687	0.0633	0.701
		4	0.0622	0.0613	0.058	0.8568

From the tables, we observe that:
1) I(-1), I(2) and I(0) decreases as a, p, q increases, and does not change as b increases, since these indexes do not depend on the scale parameter b.
2) There is no specific pattern for I(1), however I(1) increases as the parameter a increases for the chosen values of parameters p and q.

Table 6: Generalized entropy of WGB2 with $k=2$

a	p	q	I(-1)	I(2)	I(0)	I(1)
2.5	2.5	2.5	0.102	0.1335	0.0981	-0.0294
3			0.0649	0.0764	0.0633	0.0908
3.5			0.0454	0.0507	0.0446	0.1389
4			0.0337	0.0365	0.0333	0.1579
2.5	2.5	2.5	0.102	0.1335	0.0981	-0.0294
	3		0.0969	0.1293	0.0942	-0.033
	3.5		0.0931	0.1261	0.0913	-0.0357
	4		0.0903	0.1236	0.089	-0.0379
2.5	2.5	2.5	0.102	0.1335	0.0981	-0.0294
		3	0.0804	0.0922	0.0767	0.3079
		3.5	0.0687	0.0735	0.0651	0.5465
		4	0.0613	0.063	0.0579	0.7308

5 Renyi Entropy

Renyi [15] extended the concept of Shannon's entropy and is defined as follows:

$$I_\alpha(\beta) = \frac{1}{1-\beta} log\bigg(\int_0^\infty f^\beta(y)dy\bigg), \quad \beta > 0, \beta \neq 1. \tag{28}$$

Renyi entropy is important in information theory, probability and statistics as a measure of uncertainty, and tends to Shannon entropy as $\beta \to 0$.

5.1 Renyi Entropy of GB2

Recall the pdf of GB2 distribution is given by:

$$f_{GB2}(y;a,b,p,q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)[1+(\frac{y}{b})^a]^{p+q}} \quad \text{for } y>0, \text{ and } 0 \text{ otherwise}$$

Note that:

$$f^\beta_{GB2}(y;a,b,p,q) = \left(\frac{a}{bB(p,q)}\right)^\beta \left[\left(\frac{y}{b}\right)^a\right]^{\beta(p-\frac{1}{a})} \left[1+\left(\frac{y}{b}\right)^a\right]^{-\beta(p+q)},$$

where $y>0$, $a,b,p,q>0$, $\beta>0$. Let $(\frac{y}{b})^a = t$, then $dy = \frac{b}{a}t^{\frac{1}{a}-1}$, and

$$\begin{aligned}
\int_0^\infty f^\beta_{GB2}(y;a,b,p,q)dy &= \left(\frac{a}{b}\right)^{\beta-1} \frac{1}{B^\beta(p,q)} \int_0^\infty t^{\beta p - \frac{\beta}{a} + \frac{1}{a} - 1} \left(\frac{1}{t+1}\right)^{\beta(p+q)} \\
&= \left(\frac{a}{b}\right)^{\beta-1} \frac{B(\beta p - \frac{\beta}{a} + \frac{1}{a}, \beta q + \frac{\beta}{a} - \frac{1}{a} + 2)}{B^\beta(p,q)}.
\end{aligned}$$

Consequently, Renyi entropy for GB2 simplifies to:

$$I_R(\beta) = log\left(\frac{b}{a}\right) - \frac{\beta log B(p,q)}{1-\beta} + \frac{B(\beta p - \frac{\beta}{a} + \frac{1}{a}, \beta q + \frac{\beta}{a} - \frac{1}{a} + 2)}{1-\beta}, \quad (29)$$

for $a, b, p, q > 0$, $\beta > 0$, $\beta \neq 1$.

5.2 Renyi Entropy of WGB2

Recall the pdf of WGB2 distribution is given by:

$$g_{WGB2}(y; a, b, p, q, k) = \frac{a y^{ap+k-1}}{b^{ap+k} B(p + \frac{k}{a}, q - \frac{k}{a})[1 + (\frac{y}{b})^a]^{p+q}} \quad (30)$$

for $y > 0, a, b, p, q, k > 0$, and $-ap < k < aq$. Note that:

$$g_w^\beta(y; a, b, p, q, k) = \frac{(\frac{a}{b})^\beta [(\frac{y}{b})^a]^{\beta p + \frac{\beta k}{a} - \frac{\beta}{a}} [1 + (\frac{y}{b})^a]^{-\beta(p+q)}}{B^\beta(p + \frac{k}{a}, q - \frac{k}{a})}$$

for $a, b, p, q > 0$, $-ap < k < aq$, $\beta > 0$, $\beta \neq 1$. Let $(\frac{y}{b})^a = t$, then $dy = \frac{b}{a} t^{\frac{1}{a} - 1}$, and

$$\begin{aligned}\int_0^\infty g_w^\beta(y) dy &= \frac{(\frac{a}{b})^{\beta-1}}{B^\beta(p + \frac{k}{a}, q - \frac{k}{a})} \int_0^\infty \left(1 - \frac{1}{1+t}\right)^{\beta p + \frac{\beta k}{a} - \frac{\beta}{a} + \frac{1}{a} - 1} \left(\frac{1}{1+t}\right)^{\beta q - \frac{\beta k}{a} + \frac{\beta}{a} - \frac{1}{a} + 2 - 1} dt \\ &= \frac{(\frac{a}{b})^{\beta-1} B(\beta p + \frac{\beta k}{a} - \frac{\beta}{a} + \frac{1}{a}, \beta q - \frac{\beta k}{a} + \frac{\beta}{a} - \frac{1}{a} + 2)}{B^\beta(p + \frac{k}{a}, q - \frac{k}{a})}.\end{aligned}$$

Consequently, Renyi entropy for WGB2 reduces to:

$$I_R(\beta) = log\left(\frac{b}{a}\right) - \frac{\beta log B(p + \frac{k}{a}, q - \frac{k}{a})}{1-\beta} + \frac{log B(\beta p + \frac{\beta k}{a} - \frac{\beta}{a} + \frac{1}{a}, \beta q - \frac{\beta k}{a} + \frac{\beta}{a} - \frac{1}{a} + 2)}{1-\beta}, \quad (31)$$

for $a, b, p, q > 0$, $-ap < k < aq$, $\beta > 0$, $\beta \neq 1$.

We select values of the parameters a, b, p, q and compute Renyi entropy for different values of the parameter β in Tables 7 and 8.

From the tables we observe the following:
1) When $k = 1$: for $\beta < 1$, Renyi entropy increases as b, q increases; for $\beta > 1$, Renyi entropy increases as b increases, and decreases as a, p, q increases.
2) When $k = 2$: for $\beta < 1$, Renyi entropy increases as a, b increases; for $\beta > 1$, Renyi entropy increases as b increases, and decreases as a, p, q increases.

Table 7: Renyi entropy of WGB2 with $k = 1$

a	b	p	q	$\beta = 0.25$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 1.25$	$\beta = 1.5$
2.5	2.5	2.5	2.5	0.0043	0.53	4.5354	-17.6808	-13.5768
3				-0.0921	0.421	4.4616	-17.9609	-13.8119
3.5				-0.1795	0.3199	4.3792	-18.1696	-13.9942
4				-0.2599	0.2266	4.2964	-18.3358	-14.1437
2.5	2.5	2.5	2.5	0.0043	0.53	4.5354	-17.6808	-13.5768
	3			0.1866	0.7124	4.7177	-17.4985	-13.3945
	3.5			0.3407	0.8665	4.8719	-17.3444	-13.2404
	4			0.4743	1	5.0054	-17.2108	-13.1068
2.5	2.5	2.5	2.5	0.0043	0.53	4.5354	-17.6808	-13.5768
		3		-0.0843	0.5318	5.027	-19.6824	-15.0812
		3.5		-0.169	0.5261	5.4507	-21.4444	-16.4077
		4		-0.25	0.5158	5.8225	-23.0191	-17.5949
2.5	2.5	2.5	2.5	0.0043	0.53	4.5354	-17.6808	-13.5768
			3	0.1044	0.8711	5.7364	-20.3785	-15.4183
			3.5	0.1848	1.1532	6.75	-22.702	-17.0128
			4	0.251	1.3931	7.6272	-24.7466	-18.4216

6 Concluding Remarks

In this paper, the weighted generalized beta distribution of the second kind (WGB2) is presented. We showed that WGB2 includes several other distributions as special and limiting cases. The limiting and special cases include weighted generalized gamma (WGG), weighted beta of the second kind (WB2), weighted Singh- Maddala (WSM), weighted Dagum (WD), weighted gamma (WG), weighted Weibull (WW) and weighted exponential (WE) distributions as well as their unweighted or parent versions. Statistical properties of the weighted generalized beta distribution of the second kind (WGB2) including the cdf, hazard functions, monotonicity, and income-share elasticity are also presented. The moments of WGB2 as well as the mean, variance, coefficient of skewness and coefficient of kurtosis are presented. Some results on the generalized entropy and Renyi entropy of WGB2 are also presented.

ACKNOWLEDGEMENTS. The authors wish to express their gratitude to the referees and editor for their valuable comments.

Table 8: Renyi entropy of WGB2 with $k = 2$

a	b	p	q	$\beta = 0.25$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 1.25$	$\beta = 1.5$
2.5	2.5	2.5	2.5	-0.1758	0.1726	3.6929	-16.5623	-12.9325
3				-0.2436	0.1417	3.846	-17.231	-13.4088
3.5				-0.311	0.091	3.9028	-17.6653	-13.7291
4				-0.3764	0.0329	3.9124	-17.9732	-13.9639
2.5	2.5	2.5	2.5	-0.1758	0.1726	3.6929	-16.5623	-12.9325
	3			0.0065	0.355	3.8752	-16.38	-12.7502
	3.5			0.1607	0.5091	4.0294	-16.2259	-12.596
	4			0.2942	0.6427	4.1629	-16.0923	-12.4625
2.5	2.5	2.5	2.5	-0.1758	0.1726	3.6929	-16.5623	-12.9325
		3		-0.2704	0.1397	4.0278	-18.1282	-14.1235
		3.5		-0.3602	0.1046	4.3179	-19.5189	-15.1831
		4		-0.4454	0.0686	4.5734	-20.7701	-16.1377
2.5	2.5	2.5	2.5	-0.1758	0.1726	3.6929	-16.5623	-12.9325
			3	-0.0429	0.6121	5.2145	-19.9289	-15.2229
			3.5	0.0616	0.9675	6.4713	-22.7763	-17.173
			4	0.1466	1.2651	7.5436	-25.2522	-18.8767

References

[1] D. R. Cox, *Renewal Theory*, Barnes & Noble, New York, 1962.

[2] J. Esteban, Income-Share Elasticity and the Size Distribution of Income, *International Economic Review*, **27**(2), (1986), 439 - 444.

[3] R.A. Fisher, The Effects of Methods of Ascertainment upon the Estimation of Frequencies, *Annals of Human Genetics*, **6**(1), (1934), 439 - 444.

[4] R.E. Glaser, Bathtub and Related Failure Rate Characterizations, *Journal of American Statistical Association*, **75**(371), (1980), 667 - 672.

[5] R.C. Gupta and J. P. Keating, Relations for Reliability Measures Under Length Biased Sampling, *Scan. J. Statist.*, **13**(1), (1985), 49 - 56.

[6] S.P. Jenkins, Inequality and GB2 Income Distribution, *ECINQE, Society for study of Economic Inequality, Working Paper Series*, (2007).

[7] C. Kleiber and S. Kotz, Statistical Size Distributions in Economics and Actuarial Sciences, Wiley, New York, 2003.

[8] J.B. McDonald, Some Generalized Functions for the Size Distribution of Income, *Econometrica*, **52**(3), (1984), 647-663.

[9] J.B. McDonald and Y. J. Xu, A Generalization of the Beta Distribution with Application, *Journal of Econometrics*, **69**(2), (1995), 133 - 152.

[10] K.A. Nanda and K. Jain, Some Weighted Distribution Results on Univariate and Bivariate Cases, *Journal of Statistical Planning and Inference*, **77**(2), (1999), 169 - 180.

[11] B.O. Oluyede, On Inequalities and Selection of Experiments for Length-Biased Distributions, *Probability in the Engineering and Informational Sciences*, **13**(2), (1999), 129 - 145.

[12] G.P. Patil, Encountered data, statistical ecology, environmental statistics, and weighted distribution methods, *Environmetrics* , **2**(4), (1991), 377 - 423.

[13] G.P. Patil and R. Rao, Weighted Distributions and Size-Biased Sampling with Applications to Wildlife and Human Families, *Biometrics*, **34**(6), (1978), 179 - 189.

[14] C.R. Rao, On Discrete Distributions Arising out of Methods of Ascertainment, *The Indian Journal of Statistics*, **27**(2), (1965), 320 - 332.

[15] A. Renyi, On Measures of Entropy and Information, *In Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability*, **1**(1960), University of California Press, Berkeley, (1961), 547-561.

[16] C.E. Shannon, A Mathematical Theory of Communication, *The Bell System Technical Journal*, **27**(7), (1948), 379 - 423.

[17] C.E. Shannon, A Mathematical Theory of Communication, *The Bell System Technical Journal*, **27**(10), (1948), 623 - 656.

[18] M. Zelen, Problems in Cell Kinetics and Early Detection of Disease, *Reliability and Biometry*, **56**(3), (1974), 701 - 726.